Math 255A Lecture 18 Notes

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1 The Riesz-Fredholm Theorem

1.1 Conclusion of the proof of the Riesz-Fredholm theorem

Last time, we showed that if $T \in \mathcal{L}(B_1, B_2)$ with $\operatorname{im}(T)$ closed, then $\operatorname{im}(T^*) = (\ker(T))^o$ is closed, and $(B_2/\operatorname{im}(T))^* \cong \ker(T^*)$. In particular, $\dim(\operatorname{coker}(T)) = \dim(\ker(T^*))$. Apply this when $T = I + T'$, where $T' : B \to B$ is compact.

Proposition 1.1. *Let $T : B_1 \to B_2$ be compact. Then $T^* : B_2^* \to B_1^*$ is compact.*

Proof. Let $\xi_n \in B_2^*$ be bounded, $\|\xi_n\| \leq 1$ for $n = 1, 2, \dots$. Set $K = \overline{T(\{\|x\| \leq 1\}} \subseteq B_2$ compact. Consider the sequence of continuous functions $\varphi_n(x) = \langle x, \xi_n \rangle$ for $x \in K$. We have:

1. $|\varphi_n(x)| \leq \|x\| \|\xi_n\| \leq C$ for all $n = 1, 2, \dots$ and $x \in K$
2. $|\varphi_n(x) - \varphi_n(y)| = |\langle x - y, \xi_n \rangle_2| \leq \|x - y\|_1$ for $x, y \in K$.

By Ascoli's theorem, there exists a uniformly convergent subsequence (φ_{n_k}). In particular, $\sup_{\|x\| \leq 1} |\varphi_{n_k}(Tx) - \varphi_{n_\ell}(Tx)| \to 0$ as $k, \ell \to \infty$. So

$$\sup_{\|x\| \leq 1} |\langle Tx, \xi_{n_k} \rangle_2 - \langle Tx, \xi_{n_\ell} \rangle| = \sup_{\|x\| \leq 1} |\langle x, T^* \xi_{n_k} \rangle_2 - \langle x, T^* \xi_{n_\ell} \rangle| = \|T^* \xi_{n_k} - T^* \xi_{n_\ell}\|_{B_1^*} \to 0$$

as $k, \ell \to \infty$. So $(T^* \xi_{n_k})$ converges, which makes T^* compact. □

This completes our proof of the Riesz-Fredholm theorem.

Theorem 1.1 (Riesz-Fredholm)**.** *Let B be a Banach space, and let $T \in \mathcal{L}_C(B, B)$. Then $I - T$ is Fredholm, and* $\operatorname{ind}(I - T) = 0$.

Proof. Let $K : B \to B$ be compact. Then $\dim(\ker(I + K)) < \infty$, $\operatorname{im}(I + K)$ is closed, $\dim(\operatorname{coker}(I + K)) = \dim(\ker(I + K^*)) < \infty$. Thus, $I + K$ is Fredholm and $\operatorname{ind}(I + K) = \operatorname{ind}(I + \lambda K) = \operatorname{ind}(I) = 0$. □

1.2 Atkinson's theorem and stronger Riesz-Fredholm

We can actually upgrade the statement of the Riesz-Fredholm theorem to get a stronger theorem.

Proposition 1.2 (Atkinson's theorem)**.** *An operator $T \in \mathcal{L}(B_1, B_2)$ is Fredholm if and only if there exists $S \in \mathcal{L}(B_2, B_1)$ such that $TS - I$ and $ST - I$ are compact in B_2 and B_1, respectively.*

Proof. Sufficiency: Let $S \in \mathcal{L}(B_2, B_1)$ be such that $ST = I + K_1$ and $TS = I + K_2$, where K_1, K_2 are compact. Then $\ker(T) \subseteq \ker(I + K_1)$, so $|dim(\ker(T)) < \infty$. Similarly, $\operatorname{im}(T) \supseteq \operatorname{im}(I + K_2)$, so $\dim(\operatorname{coker}(T)) \leq \dim(\operatorname{coker}(I + K_2)) < \infty$. So T is Fredholm.

Necessity: Take the Grushin approach: if T is Fredholm, write $n_+ = \dim(\ker(T))$, and $n_- = \dim(\operatorname{coker}(T))$. Then there exist an injective $R_- : \mathbb{C}^{n_-} \to B_2$ with $B_2 = \operatorname{im}(T) \oplus R_-(\mathbb{C}^{n_-})$ and a surjective $R_+ : B_1 \to \mathbb{C}^{n_+}$ such that $R_+|_{\ker(T)}$ is bijective. Then the operator $\mathcal{P} : B_1 \oplus \mathbb{C}^{n_-} \to B_2 \oplus \mathbb{C}^{n_+}$ given by

$$\mathcal{P} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix}$$

is invertible. It has the inverse

$$\mathcal{E} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix}.$$

We get that

$$\mathcal{P}\mathcal{E} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} TE + R_-E_- & * \\ * & * \end{bmatrix},$$

$$\mathcal{E}\mathcal{P} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} = \begin{bmatrix} ET + E_+R_+ & * \\ * & * \end{bmatrix},$$

so $TE - I = -R_0E$ and $ET - I = -E_+R_+$. The maps R_-E_- and E_+R_+ have finite rank, so they are compact. □

Remark 1.1. If $ET - I$ and $TE - I$ are compact, then $E \in \mathcal{L}(B_2, B_1)$ is Fredholm, and $\operatorname{ind}(ET) = \operatorname{ind}(I + K) = 0$. By the logarithmic law, this equals $\inf(E) + \operatorname{ind}(T)$. So $\operatorname{ind}(E) = -\operatorname{ind}(T)$.

Theorem 1.2. *Let $T \in \mathcal{L}(B_1, B_2)$ be Fredholm and $S \in \mathcal{L}_C(B_1, B_2)$. Then $T + S$ is Fredholm, and* $\operatorname{ind}(T + S) = \operatorname{ind}(T)$.

Proof. Let $E \in \mathcal{L}(B_2, B_1)$ be such that $TE - I$ and $ET - I$ are compact. Then $(T+S)E - I$ and $E(T+S) - I$ are compact. So $T + S$ is Fredholm. Also, $\operatorname{ind}(T+S) = \operatorname{ind}(T + \lambda S)$ for $\lambda \in \mathbb{C}$. Letting $\lambda \to 0$, we get $\operatorname{ind}(T+S) = \operatorname{ind}(T)$. □

1.3 Applications to differential equations

Proposition 1.3. *Let $a, b \in C([0,1])$, and consider the boundary value problem $u'' + au + bu = f$ with boundary conditions $u(0) = u(1) = 0$. Here, $f \in C([0,1])$, and $u \in C^2([0,1])$. The boundary value problem has a unique solution for any $f \in C([0,1])$ if and only if the homogeneous problem when $f = 0$ only has the trivial solution.*

Proof. Let $B_1 = \{u \in C^2([0,1]) : u(0) = u(1) = 0\}$ be a Banach space. Let $B_2 = C([0,1])$ with

$$\|u\|_{B_2} = \sum_{j=0}^{2} \|u^{(j)}\|_{L^\infty}.$$

Then $T : B - 1 \to B_2$ sending $u \mapsto u''$ is bijective, so $\operatorname{ind}(T) = 0$. The map $S : B_1 \to B_2$ sending $u + au' + bu$ is compact, so $T + S$ is Fredholm with $\operatorname{ind}(T + S) = 0$. So $T + S$ is bijective if and only if $T + S$ is injective. $\square$

Our next application will be the Toeplitz index theorem. Here is the idea. Let H be the closed subspace of $L^2(\mathbb{R}/2\pi\mathbb{Z})$ with Fourier coefficients $\hat{u}(n) = 0$ for $n < 0$. Let $\Pi : L^2 \to H$ be an orthogonal projection. Let $f \in C(\mathbb{C}/2\pi\mathbb{Z}) \mapsto \operatorname{Top}(f)$, which sends $u \mapsto \Pi(fu)$. This is called the **Toeplitz operator**.

Theorem 1.3. $\operatorname{Top}(f)$ *is Fredholm if and only if $f \neq 0$. Moreover,* $\operatorname{ind}(\operatorname{Top}(f)) = -$*winding number of f.*